Modelling the inflation of incompressible, hyperelastic shells

Special Topic report for the *C5.1 Solid Mechanics* course taught by Prof. Dominic Vella

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Contents

1 Introduction

This report pertains to the analysis of balloon instability. It starts, in Section 2, with a qualitative introduction to the phenomenon for a general cylindrical shell. Then, before the main problem, the general techniques are introduced in Section 3 with the analysis of the inflation of a spherical shell. Then, in Section 4 the general cylindrical shell model is defined, simplified and analysed. Then, Section 5 pertains to the analysis required to find the characteristic pressure values of the inflation.

2 Qualitative explanation

When a cylindrical hyperelastic shell (such as a long party balloon) is inflate by slowly filling it with air, it can contain sections in two distinct states, one with much larger strain than the other, while there still is a spatially constant internal pressure. This section gives a qualitative explanation of the process, through Figure 1, based on [1, 5, 8]. The top row of the figure shows the shape of the balloon for each state, the middle row the position(s) on the pressure P versus stretch λ_a relationship derived later, and the bottom row the pressure P versus volume V history of the inflationary process.

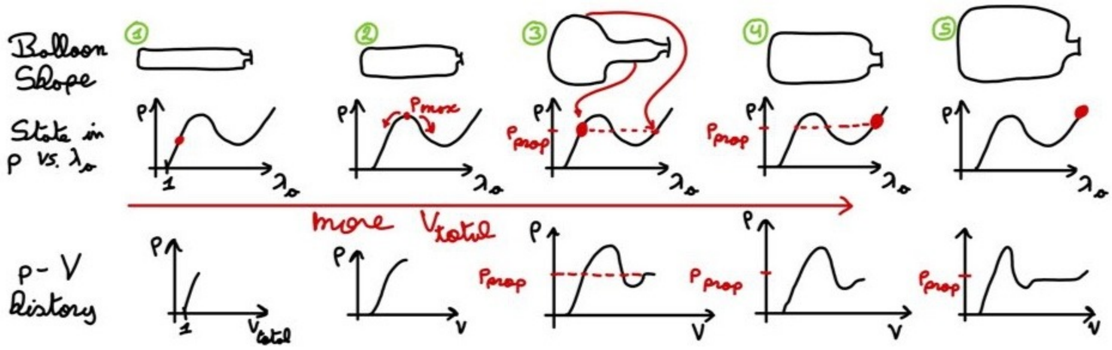

Figure 1: Progression of balloon inflation

In state 1, the internal pressure is above external pressure, and the strain increases (non-linearly but monotonously) with the pressure. Higher pressure, higher deformation, everywhere.

Each section of the balloon has a specific pressure, defined P_{max}, above which the deformation changes behaviour. Then, depending on how the inflation is carried out, the deformation proceeds differently. If the inflation process is pressure specified, then crossing P_{max} would result in a jump in the deformation, directly to the higher strain at the same pressure. The other option, which we will henceforth assume, is that an air mass flow rate (i.e. adding 10 g/s of air to the balloon) is specified. Thus we specify a (time dependent) volume, and measure the pressure and strain.

As the air mass increases, a certain weaker section will have its local P_{max} exceeded and start deforming much faster than the rest (consider the P versus λ_a graph of state 2 in Figure 1). This large expansion constitutes an increase in volume, whereby, under quasi steady-state conditions, the internal pressure drops. This results in only one section exceeding its P_{max} and thus only that section transitions into a second stable state for that pressure, the bulge has then localized. Note that while there may be three strains that correspond to a specific pressure, the one with the negative slope is unstable as any material section will simply "strain away from it".

The result, state 3, is a balloon in two distinct, stable phases with a smooth boundary between them (still with a spatially constant gas pressure). The phase boundary between these two states is quite complex due to the non-homogeneous stretch and not considered here. If gas is then added, a low strain section of the balloon near the phase boundary transforms into a high strain section, effectively moving the boundary, state 4. Once the entire balloon is in the high strain phase, homogeneous-strain expansion resumes, shown as state 5.

Deflation The models that we are about to derive, and the explanation above only cover the inflation of a balloon, not the deflation. It is distinct from simply being inflation in reverse, for a number of reasons:

1. The material, when deformed to the high-strain state, may undergo non-reversible physical changes, such as plastic deformation. Even if the changes are reversible, the deformation process may be different or poorly modelled by the chosen strain energy function.

2. The path taken on the P versus λ_a curve is non-trivial. Decreasing the air mass from state 5, would bring material sections to the local minimum, from where it is not directly clear what would happen if the air mass is further decreased.

3 Model for a spherical shell

In this section, we will derive the behaviour of a spherical (as opposed to cylindrical) shell, as a simplification of the general case qualitatively explained in Section 2, based on the lectures by Prof. Dominic Vella and the corresponding lecture notes [8, p. 48].

This will introduce the solving technique and illustrate the effect of the strain energy function. Due to its symmetry, a spherical shell will not exhibit the low and high strain phases simultaneously, but we can still recover the P versus λ_a curve.
The main assumptions made while deriving the model are:

- We consider all transformations to occur under quasi steady-state conditions such that there is no time dependent behaviour.
- The shell is made of an incompressible, hyperelastic material, then the following relationship holds: $\boldsymbol{T} = J^{-1}\boldsymbol{F}\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}} - \tilde{p}\mathbf{1}$, with $J = \det(\boldsymbol{F}) = 1$
- The only radial stress on the two boundary surfaces is the pressure.

3.1 Kinematics

Consider the deformation of a spherical shell from its initial state to the current state shown in Figure 2. This subsection defines the deformation, and finds the corresponding mapping $\boldsymbol{Q} \to \boldsymbol{q}$ and deformation gradient.

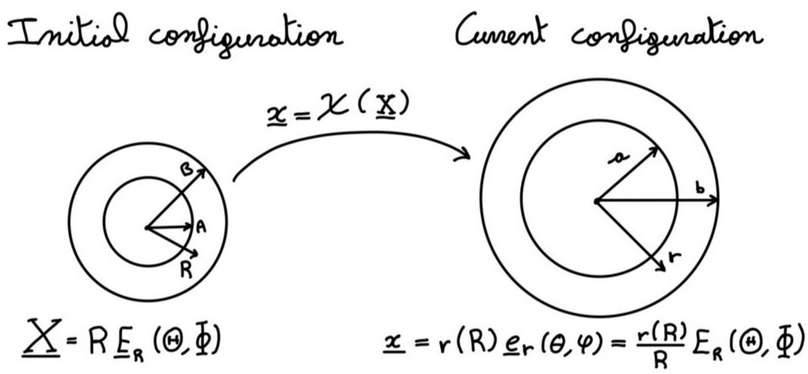

Figure 2: Deformation of a spherical balloon

Defining the transformation The deformation is defined such that, for any given point, only the distance to the origin changes, (3.1). Points in the original configuration are described by $\boldsymbol{Q} = \{R, \Theta, \Phi\}$, while in the current configuration they are given in terms of $\boldsymbol{q} = \{r, \theta, \varphi\}$. These cylindrical coordinates can be transformed into the Cartesian reference frame using (3.2).

$$\boldsymbol{q}(\boldsymbol{Q}) = \begin{bmatrix} r(R) \\ \theta(\Theta) \\ \varphi(\Phi) \end{bmatrix} = \begin{bmatrix} r(R) \\ \Theta \\ \Phi \end{bmatrix} \tag{3.1}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r\cos(\varphi)\sin(\theta) \\ r\sin(\varphi)\sin(\theta) \\ r\cos(\theta) \end{bmatrix} \qquad \boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} R\cos(\Phi)\sin(\Theta) \\ R\sin(\Phi)\sin(\Theta) \\ R\cos(\Theta) \end{bmatrix} \tag{3.2}$$

$$\boldsymbol{e}_\alpha = h_\alpha^{-1}\frac{\partial \boldsymbol{x}}{\partial q_\alpha}, \qquad \boldsymbol{E}_\beta = H_\beta^{-1}\frac{\partial \boldsymbol{X}}{\partial Q_\beta} \qquad \alpha, \beta = 1, 2, 3 \tag{3.3}$$

$$h_\alpha = \left|\frac{\partial \boldsymbol{x}}{\partial q_\alpha}\right| \qquad H_\beta = \left|\frac{\partial \boldsymbol{X}}{\partial Q_\beta}\right| \tag{3.4}$$

Finding the deformation gradient Each coordinate has an associated basis vector, defined from the Cartesian vector, (3.3), where the h_α and H_β are scale factors as defined in (3.4) [8, p. 17]. The deformation gradient $\boldsymbol{F}$ then becomes [8, p. 20]:

$$\begin{aligned} \boldsymbol{F} &= \mathrm{Grad}(\boldsymbol{x}(\boldsymbol{X})) \\ &= H_\beta^{-1}\frac{\partial \boldsymbol{x}}{\partial Q_\beta} \otimes \boldsymbol{E}_\beta \\ &= H_\beta^{-1}\left(\frac{\partial \boldsymbol{x}}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial Q_\beta}\right) \otimes \boldsymbol{E}_\beta \\ &= H_\beta^{-1}\left(h_\alpha \boldsymbol{e}_\alpha \frac{\partial q_\alpha}{\partial Q_\beta}\right) \otimes \boldsymbol{E}_\beta \\ \boldsymbol{F} &= \frac{h_\alpha}{H_\beta}\frac{\partial q_\alpha}{\partial Q_\beta}\boldsymbol{e}_\alpha \otimes \boldsymbol{E}_\beta. \end{aligned} \tag{3.5}$$

For a spherical coordinate system, the scale factors are:

$$h_r = 1 = H_r, \qquad h_\theta = r \to H_\Theta = R, \qquad h_\varphi = r\sin(\theta) \to H_\Phi = R\sin(\Theta). \tag{3.6}$$

Then, using these scale factors, the definition of the transform (3.1), and the modified transformation gradient equation (3.5), we can find it to be :

$$\boldsymbol{F} = \frac{\mathrm{d}r}{\mathrm{d}R}\boldsymbol{e_r} \otimes \boldsymbol{E_R} + \frac{r}{R}\boldsymbol{e_\theta} \otimes \boldsymbol{E_\Theta} + \frac{r\sin(\theta)}{R\sin(\Theta)}\boldsymbol{e_\varphi} \otimes \boldsymbol{E_\Phi}. \tag{3.7}$$

Then, noting that $\theta = \Theta$, we can write the deformation gradient in matrix form:

$$[\boldsymbol{F}] = \begin{bmatrix} r'(R) & & \\ & r/R & \\ & & r/R \end{bmatrix}. \tag{3.8}$$

As the diagonal entries of $[\boldsymbol{F}]$ are the derivatives between corresponding coordinate vectors of the initial and current configuration, they are called principle stretches. Let $\lambda_r = r'(R)$, $\lambda_\theta = r/R = \lambda_\phi$.

Explicit deformation Next, we may apply the assumption of an incompressible material: $J = \det(\boldsymbol{F}) = 1$:

$$r'(R) \cdot \frac{r^2}{R^2} = 1 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{r^3}{3}\right) = R^2 \quad \rightarrow \quad \frac{r^3}{3} = \frac{R^3}{3} + C_1. \tag{3.9}$$

To fix the unknown constant C_1, we can apply a boundary condition: $r(A) = a$:

$$r(A)^3 = a^3 = A^3 + 3C_1 \quad \rightarrow \quad r(R) = \sqrt[3]{a^3 - A^3 + R^3}. \tag{3.10}$$

Note that only one BC could be applied (not the one at $r(B) = b$). Due to the assumption of incompressibility, the strain on one boundary, $r = a$ in this case, fully determines the transformation.

3.2 Dynamics

Having an explicit expression for the deformation and the deformation gradient, we can now look at how this deformation relates to the internal stress in the material.

Cauchy's first equation of motion The general form of Cauchy's first equation of motion [8, p. 30] reduces, under the assumption of no body forces ($\boldsymbol{b} = 0$), and no time dependent behaviour ($\partial/\partial t \rightarrow 0$), to $\mathrm{div}(\boldsymbol{T}) = 0$. To find an expression for $\boldsymbol{T}$, we use the following constitutive law, which for the diagonal $\boldsymbol{F}$ yields:

$$\boldsymbol{T} = \boldsymbol{F}\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}} - \tilde{p}\mathbf{1} = \begin{bmatrix} \lambda_r & & \\ & \lambda_\theta & \\ & & \lambda_\varphi \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}W}{\mathrm{d}\lambda_r} & & \\ & \frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} & \\ & & \frac{\mathrm{d}W}{\mathrm{d}\lambda_\varphi} \end{bmatrix} - \tilde{p} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}. \tag{3.11}$$

Then:

$$T_{rr} = \lambda_r \frac{\mathrm{d}W}{\mathrm{d}\lambda_r} - \tilde{p}, \qquad T_{\theta\theta} = \lambda_\theta \frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} - \tilde{p}, \qquad T_{\varphi\varphi} = \lambda_\varphi \frac{\mathrm{d}W}{\mathrm{d}\lambda_\varphi} - \tilde{p} \tag{3.12}$$

where $\tilde{p}$ is an unknown constant (different from the internal pressure P), and all other entries are 0. Then, $\mathrm{div}(\boldsymbol{T})$ (expanded for spherical coordinates in [3, p 121]), reduces to:

$$\operatorname{div}(\boldsymbol{T}) = \left(\frac{\partial T_{rr}}{\partial r} + 2\frac{T_{rr}}{r} - \frac{1}{r}(T_{\theta\theta} + T_{\varphi\varphi})\right)\boldsymbol{e_r} \qquad (3.13)$$
$$+ \left(\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{\cot(\theta)}{r}(T_{\theta\theta} - T_{\varphi\varphi})\right)\boldsymbol{e_\theta}$$
$$+ \left(\frac{1}{r\sin(\theta)}\frac{\partial T_{\varphi\varphi}}{\partial\varphi}\right)\boldsymbol{e_\varphi}.$$

The entry in the $\boldsymbol{e_r}$ direction, noting $\lambda_\theta = \lambda_\phi$, then reduces to:

$$\begin{aligned} 0 &= \frac{\partial T_{rr}}{\partial r} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) \\ &= \frac{\partial T_{rr}}{\partial r} + \frac{2}{r}\left(\lambda_r\frac{\mathrm{d}W}{\mathrm{d}\lambda_r} - \lambda_\theta\frac{\mathrm{d}W}{\mathrm{d}\lambda\theta}\right). \qquad (3.14) \end{aligned}$$

Applying symmetry Using the symmetry in the problem, (3.14) can be reduced to an Ordinary Differential Equation. First we notice that since $\lambda_\theta = \lambda_\phi = \lambda$ and $J = 1$, the principal stretches reduce to just one independent variable:

$$\lambda_r = 1/\lambda^2, \qquad \lambda_\theta = \lambda, \qquad \lambda_\phi = \lambda$$

Therefore, the strain function, also just depends one on variable, such that we may define:

$$h(\lambda) = W(\lambda_r = 1/\lambda^2, \lambda_\theta = \lambda, \lambda_\phi = \lambda)$$

Then we notice that:

$$\begin{aligned} \frac{\mathrm{d}h(\lambda)}{\mathrm{d}\lambda} &= \frac{\mathrm{d}W}{\mathrm{d}\lambda} \\ &= \frac{\mathrm{d}W}{\mathrm{d}\lambda_r}\frac{\mathrm{d}\lambda_r}{\mathrm{d}\lambda} + \frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta}\frac{\mathrm{d}\lambda_\theta}{\mathrm{d}\lambda} + \frac{\mathrm{d}W}{\mathrm{d}\lambda_\phi}\frac{\mathrm{d}\lambda_\phi}{\mathrm{d}\lambda} \\ &= \frac{\mathrm{d}W}{\mathrm{d}\lambda_r}\frac{-2}{\lambda^3} + 2\frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} \\ -\frac{\lambda}{2}\frac{\mathrm{d}h}{\mathrm{d}\lambda} &= \left(\frac{1}{\lambda^2}\right)\frac{\mathrm{d}W}{\mathrm{d}\lambda_r} - \lambda\frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} \end{aligned}$$

which is exactly the term in the brackets in (3.14). Next, we turn to the partial derivative:

$$\begin{aligned} \frac{\partial T_{rr}}{\partial r} &= \frac{\partial T_{rr}}{\partial\lambda}\frac{\partial}{\partial r}\left(\frac{r}{R(r)}\right) = \frac{\partial T_{rr}}{\partial\lambda}\left(\frac{1}{R} - rR^{-2}R'\right) \\ &= \frac{\partial T_{rr}}{\partial\lambda}\left(\frac{1}{R} - \frac{r}{R^2}\cdot r^2R^{-2}\right) = \frac{\partial T_{rr}}{\partial\lambda}\frac{1 - r/R\cdot(r^2/R^2)}{R} \\ &= \frac{\partial T_{rr}}{\partial\lambda}\frac{1-\lambda^3}{R} \end{aligned}$$

where we applied (3.10).

Solving the ODE With the transformations done above, (3.14) becomes:

$$\frac{\partial T_{rr}}{\partial \lambda} = \frac{1}{1-\lambda^3}\frac{\mathrm{d}h}{\mathrm{d}\lambda} \tag{3.15}$$

which is an ODE in λ for T_{rr}, necessitating boundary conditions. Consider Figure 3, by assuming the only applied stress on the boundary is due to the pressure, P, and remembering $\boldsymbol{t_n} = \boldsymbol{Tn}$, we find:

$$\boldsymbol{t_n} = \begin{cases} -P\boldsymbol{n} & r = r_a \\ 0 & r = r_b \end{cases} \quad \rightarrow \quad T_{rr} = \begin{cases} -P & r = r_a \\ 0 & r = r_b \end{cases} \tag{3.16}$$

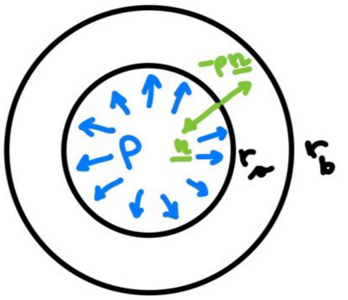

Figure 3: Boundary conditions spherical shell with internal pressure P

Note that the minus sign is because the pressure acts in the direction opposite to the surface normal, which is consistent with the convention that stress which causes compression should be negative. Thus we have the known radial stress on the surfaces of the shell. With that, we may integrate (3.15) to find:

$$T_{rr}(\lambda) = T_{rr}(\lambda_b) + \int_{\lambda_b}^{\lambda} \frac{h'(\lambda)}{1-\lambda^3}\mathrm{d}\lambda$$

$$T_{rr}(\lambda_a) = -P = 0 + \int_{\lambda_b}^{\lambda_a} \frac{h'(\lambda)}{1-\lambda^3}\mathrm{d}\lambda$$

$$P = \int_{\lambda_a}^{\lambda_b} \frac{h'(\lambda)}{1-\lambda^3}\mathrm{d}\lambda \tag{3.17}$$

where $\lambda_a = r_a/R_a$ and $\lambda_b = r_b/R_b$. Note that in the last step the order of integration was reversed, now from the inner to the outer boundary. To make further analytical progress, we must choose a strain energy function.

Neo-Hookian strain energy function The simplest strain energy function, the Neo-Hookian, becomes, for this deformation:

$$W_{NH} = \frac{\mu}{2}(I_1 - 3) = \frac{\mu}{2}(\lambda_r^2 + \lambda_\theta^2 + \lambda_\phi^2 - 3) = \frac{\mu}{2}(2\lambda^2 + 1/\lambda^4 - 3) \equiv h(\lambda). \tag{3.18}$$

Thus
$$h'(\lambda) = \frac{\mu}{2}(4\lambda - 4\lambda^{-5}) = 2\mu\frac{\lambda^6-1}{\lambda^5} = 2\mu\frac{(\lambda^3-1)(\lambda^3+1)}{\lambda^5}.$$
Yielding the final expression for the pressure:
$$P = -2\mu\int_{\lambda_a}^{\lambda_b}\frac{\lambda^3+1}{\lambda^5}\mathrm{d}\lambda = -2\mu\left[\frac{1}{\lambda}+\frac{1}{4\lambda^4}\right]_{\lambda_a}^{\lambda_b} \tag{3.19}$$
where $\lambda_b(\lambda_a) = \sqrt[3]{1 + A^3/B^3(\lambda_a^3 - 1)}$, derived from (3.10). Plotting the pressure versus the stretch λ_a yields Figure 4. Notice that for a stretch under ≈ 2, increasing pressure increases stretch. However, once the critical pressure is exceeded, the balloon keeps straining indefinitely. This trend does not change as the equation is always positive ($\lambda_a < \lambda_b$) and decreasing (negative powers). This is an example of inflationary instability. This behaviour is due to the choice of strain energy function, and not a physical phenomenon, contradicting Figure 1.

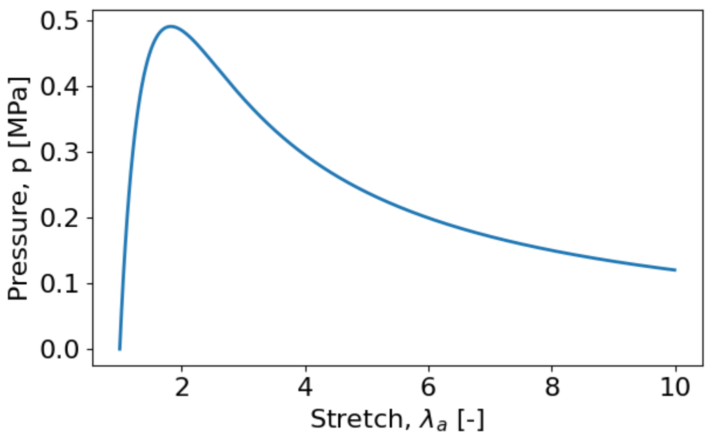

Figure 4: Graph of the pressure vs stretch relationship of a spherical shell for a Neo-Hookian material with $R \in [1, 2]$m and shear modulus of 0.6MPa (reasonable estimation for natural rubber [2]).

Mooney-Rivlin strain energy function As a hopefully more realistic strain energy function let us now consider Mooney-Rivlin, which becomes:
$$W_{MR} = \frac{C_1}{2}(I_1 - 3) + \frac{C_2}{2}(I_2 - 3) = \frac{C_1}{2}(2\lambda^2 + 1/\lambda^4 - 3) + \frac{C_2}{2}(2/\lambda^2 + \lambda^4 - 3) \equiv h(\lambda) \tag{3.20}$$
Then
$$\begin{aligned} h'(\lambda) &= 2C_1(\lambda - \lambda^{-5}) + 2C_2(\lambda^3 + \lambda^{-3}) \\ &= 2C_1\frac{(\lambda^3-1)(\lambda^3+1)}{\lambda^5} + 2C_2\frac{(\lambda^3-1)(\lambda^3+1)}{\lambda^3} \end{aligned}$$

such that the pressure-strain relationship is:

$$P = \left[C_1 \left(\frac{2}{\lambda} + \frac{1}{2\lambda^4} \right) + C_2 \left(\frac{1}{\lambda^2} - 2\lambda \right) \right]_{\lambda_a}^{\lambda_a(\lambda_b)} \tag{3.21}$$

which can once again be plotted, as shown in Figure 5. From the positive gradient for $\lambda > 5$, we conclude that this strain energy does not exhibit inflationary instability, and in fact looks very similar to the expected behaviour from Figure 1.

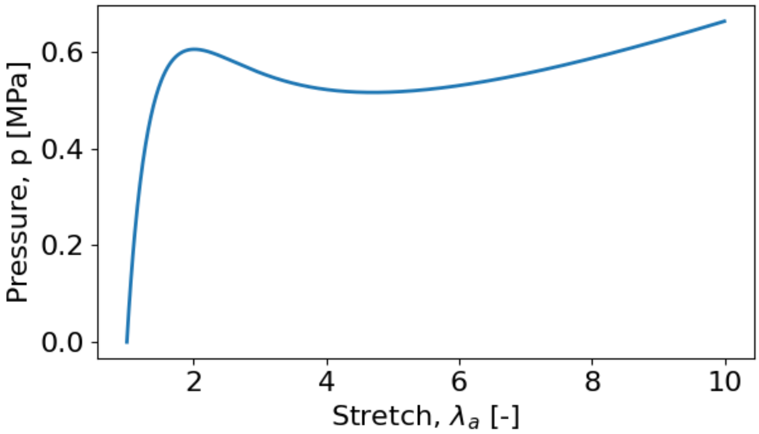

Figure 5: Pressure versus stretch relationship of a spherical shell for a Mooney-Rivlin strain energy function with $R \in [1, 2]$m and $C_1 = 0.621$MPa and $C_2 = 0.054$MPa (reasonable estimation for rubber [6]) The code for generating this and Figure 4 is provided in Appendix B

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Van Gent and Ogden-1 strain energy functions Performing the same analysis for the van Gent strain energy function

$$W_{GE} = \frac{-\mu}{2\beta} \ln\left(1 - \beta(I_1 - 3)\right) = -\frac{\mu}{2\beta} \ln\left(1 - \beta(2\lambda^2 + \lambda^{-4} - 3)\right) \tag{3.22}$$

and the Ogden-1 strain energy function

$$W_{OG_1} = \frac{2\mu}{\beta^2} \left(\lambda_1^\beta + \lambda_2^\beta + \lambda_3^\beta - 3 \right) = \frac{2\mu}{\beta^2} \left(2\lambda^\beta + \lambda^{-2\beta} - 3 \right). \tag{3.23}$$

leads to the following pressure versus stretch relationships:

$$\text{Van Gent}: \quad P = -2\mu \int_{\lambda_a}^{\lambda_b} \frac{\lambda^2 + \lambda^{-5}}{(1 - \beta(2\lambda^2 + \lambda^{-4} - 3))} \mathrm{d}\lambda \tag{3.24}$$

$$\text{Ogden-1}: \quad P = 4\mu \int_{\lambda_a}^{\lambda_b} \frac{\lambda^{\beta-1} - \lambda^{-2\beta-1}}{(1 - \lambda^3)\beta} \mathrm{d}\lambda \tag{3.25}$$

The computation of these integrals for general β is quite complicated and beyond the scope of this report.

The analysis in this section illustrates how we can describe a deformation (kinematics), apply rules such as Cauchy 1 (dynamics), and then find an expression for the pressure inside a hyperelatic spherical shell. However, then, the selection of a strain energy function causes the deterministic solution process to break down: there is no longer a single correct answer. The Neo-Hookian strain energy function, while it does not exhibit the dual phase behaviour, is applicable for small deformations. The Mooney Rivlin function is not well suited for strains above 100% [7], for which the Ogden strain energy function is more appropriate. However, Ogden-N requires more parameters to be fit to the material. In summary, there is uncertainty regarding the applicability all of these results that physical tests are best suited to resolve.

4 Model for a cylindrical shell

With the spherical case considered, we now proceed to the (more complicated) cylindrical case. Consider a long, circular, incompressible, elastic tube, with a varying radius along its length. On both ends, flat endcaps are applied, which, while difficult to apply in practice, are compatible with a uniform axial expansion, just as [4]. This tube has an internal pressure and an axially applied force. The aim of this derivation is to find a relationship between the deformation of the tube (in both radial and axial directions) and the applied force F and pressure P. We will first describe the general problem, then analyse a simplified model. The assumptions made in Section 3 for the spherical shell are also made for this case.

4.1 Kinematics

The initial configuration is described by the cylindrical coordinates $\boldsymbol{Q} = \{R, \Theta, Z\}$, and in the current configuration by $\boldsymbol{q} = \{r, \theta, z\}$, as shown in Figure 6. Each coordinate set has an associated set of basic vectors, (3.3).

To find the gradient of this transformation, $\boldsymbol{F}$, we first characterize the deformation. For this, we need the transformation from cartesian to cylindrical coordinates (4.26), and the relationship between the initial and current configurations (4.27).

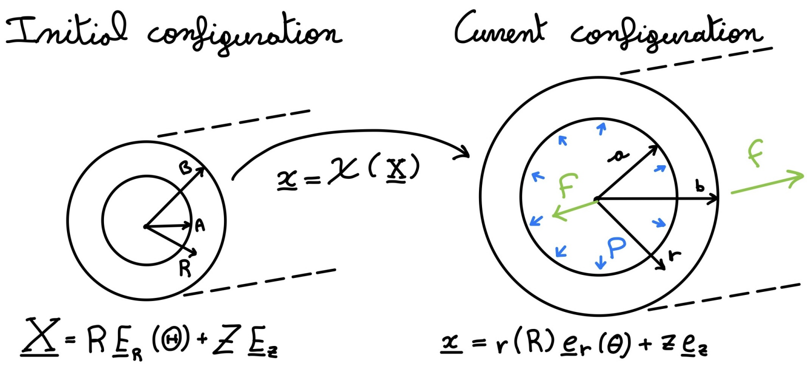

Figure 6: Cylindrical shell geometry where F is the applied axial force and P is the internal pressure.

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} R\cos(\Theta) \\ R\sin(\Theta) \\ Z \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{bmatrix} \tag{4.26}$$

$$\boldsymbol{q}(\boldsymbol{Q}) = \begin{bmatrix} r \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} r(R,Z) \\ \Theta \\ z(Z) \end{bmatrix} \tag{4.27}$$

Using (4.26) and (3.4) to evaluate the scale factors of (3.5), and (4.27) for the derivative term, we find:

$$\boldsymbol{F} = \frac{\partial r}{\partial R}\boldsymbol{e}_r \otimes \boldsymbol{E}_R + \frac{\partial r}{\partial Z}\boldsymbol{e}_r \otimes \boldsymbol{E}_Z + \frac{r}{R}\boldsymbol{e}_\theta \otimes \boldsymbol{E}_\Theta + \frac{\mathrm{d}z}{\mathrm{d}Z}\boldsymbol{e}_z \otimes \boldsymbol{E}_Z. \tag{4.28}$$

The resulting deformation gradient is not diagonal and the presence of a second (unknown) derivative term, greatly complicates the analysis. Let's explore simplifying assumptions.

Thin-walled model We could assume the wall to be so thin ($B - A \ll A$) that the wall becomes a surface with constant thickness, such that $r(R,Z)$ reduces to $r(Z)$, which is the simplification made by C. Lestringant and B. Audoly in [5]. However, we will go a different route.

4.2 Simplified model

If we assume that the deformation is homogeneous along the length of the tube, two changes happen: 1) r only depends on R ($r(R)$), and 2) the axial stretch is constant:

$z = \zeta Z$. The first part of this subsection follows [8, p. 52]. This does reduce the range of applicability of the model: it is not suitable for describing the shape of the transition between the low and high strain sections.

Kinematics The deformation gradient then simplifies to:

$$\boldsymbol{F} = \frac{\mathrm{d}r(R)}{\mathrm{d}R}\boldsymbol{e}_r \otimes \boldsymbol{E}_R + \frac{r}{R}\boldsymbol{e}_\theta \otimes \boldsymbol{E}_\theta + \zeta \boldsymbol{e}_z \otimes \boldsymbol{E}_Z \tag{4.29}$$

whereby $\boldsymbol{F}$ is diagonal. Therefore, the principal stretches are the diagonal entries of $\boldsymbol{F}$:

$$\lambda_r = \frac{\mathrm{d}r(R)}{\mathrm{d}R}, \quad \lambda_\theta = \frac{r}{R} \equiv \lambda, \quad \lambda_z = \zeta$$

Next, we assumed an isotropic material, which implies:

$$J = \lambda_r \lambda_\theta \lambda_z = 1 \quad \rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}R} r = \frac{R}{\zeta} \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{r^2}{2}\right) = \frac{R}{\zeta}$$

Noting that a boundary condition for the transformation is $r(A) = a$ we can then solve the ODE:

$$\frac{r^2}{2} = \frac{R^2}{2\zeta} + C \quad \rightarrow \quad r(R) = \sqrt{a^2 + \frac{R^2 - A^2}{\zeta}} \tag{4.30}$$

We may then conclude that $\lambda = r(R)/R$, and hence also the deformation, is fully determined by the initial state (the parameter A) and the parameters $\lambda_a = a/A$ and ζ. For example, the strain of the outer wall radius, $\lambda_b = b/B$, can then be written as:

$$\lambda_b = \frac{r(B)}{B} = \sqrt{\frac{a^2}{B^2} + \frac{B^2 - A^2}{B^2\zeta}} = \sqrt{\frac{\lambda_a^2 A^2}{B^2} + \frac{1}{\zeta} - \frac{A^2}{B^2\zeta}} = \sqrt{\frac{1}{\zeta} + \frac{A^2}{B^2\zeta}(\zeta\lambda_a^2 - 1)}$$

Dynamics Just as for the spherical shell, we may infer from the diagonal $\boldsymbol{F}$ that $\boldsymbol{T}$ is diagonal too, with values:

$$[\boldsymbol{T}] = \mathrm{diag}\left(T_{rr} = \lambda_r \frac{\mathrm{d}W}{\mathrm{d}\lambda_r} - \tilde{p},\ T_{\theta\theta} = \lambda_\theta \frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} - \tilde{p},\ T_{zz} = \lambda_z \frac{\mathrm{d}W}{\mathrm{d}\lambda_z} - \tilde{p}\right) \tag{4.31}$$

The first Cauchy equation [8, p. 30] with no body forces ($\boldsymbol{b} = 0$) and steady state ($\partial/\partial t \rightarrow 0$) reduces to $\mathrm{div}(\boldsymbol{T}) = 0$. For a diagonal $\boldsymbol{T}$ in cylindrical coordinates [3, p 121] this becomes:

$$\boldsymbol{0} = \left[\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}(T_{rr} - T_{\theta\theta})\right]\boldsymbol{e}_r + \left[\frac{1}{r}\frac{\partial T_{rr}}{\partial r}\right]\boldsymbol{e}_\theta + \left[\frac{\partial T_{zz}}{\partial z}\right]\boldsymbol{e}_z \tag{4.32}$$

where the $\boldsymbol{e}_r$ component is called the balance law. Integrating it yields:

$$T_{rr}(r) = T_{rr}(a) + \int_a^r \frac{T_{\theta\theta} - T_{rr}}{r} \mathrm{d}r. \tag{4.33}$$

Next, we prescribe the same boundary conditions as for the spherical shell: $T_{rr}(a) = -P$, $T_{rr}(b) = 0$, yielding:

$$P = \int_a^b \frac{T_{\theta\theta} - T_{rr}}{r} \mathrm{d}r \tag{4.34}$$

which provides the pressure required to achieve a certain stress state. This is one equation, but as we have two unknowns, we need a second one. It comes from force equilibrium in the endcaps. This could take the form $T_{zz}(z = 0) = F_{total} = T_{zz}(z = h)$, but there would be a dependence on r. Therefore, we use an integral condition, integrating over an annulus from a to b:

$$\int_a^b T_{zz}(r) 2\pi r \mathrm{d}r = F_{total} = F + P\pi a^2 \tag{4.35}$$

where $T_{zz}(r)$ is the axial stress (with unit Pa = N/m^2), and F_{total} is the total axial force, which is the sum of the externally applied force F and the pressure over the internal area of the endcap $P \cdot \pi a^2$. However, for incompressible materials this is not convenient due to the presence of $\tilde{p}$ in (4.31). Hence we shall transform the integral:

$$2\pi \int_a^b T_{zz} r \mathrm{d}z = 2\pi \int_a^b (T_{zz} - T_{rr} + T_{rr}) r \mathrm{d}r$$

Then we use integration by parts for T_{rr}, and the balance law (4.32):

$$\begin{aligned} 2\pi \int_a^b T_{rr} r dr &= 2\pi \left[T_{rr} \cdot \frac{r^2}{2} \right]_a^b - 2\pi \int_a^b \frac{\partial T_{rr}}{\partial r} \frac{r^2}{2} \mathrm{d}r \\ &= T_{rr}(b) b^2 \pi - T_{rr}(a) a^2 \pi - \pi \int_a^b \frac{1}{r} (T_{\theta\theta} - T_{rr}) r^2 \mathrm{d}r \\ &= P a^2 \pi - \pi \int_a^b (T_{\theta\theta} - T_{rr}) r \mathrm{d}r \end{aligned}$$

Then

$$2\pi \int_a^b (T_{zz} - T_{rr} + T_{rr}) r \mathrm{d}r = \pi \int_a^b (2T_{zz} - 2T_{rr} - (T_{\theta\theta} - T_{rr})) r \mathrm{d}r + P a^2 \pi$$

which can be combined with (4.35) to find:

$$F = \pi \int_a^b (2T_{zz} - T_{rr} - T_{\theta\theta}) r \mathrm{d}r \tag{4.36}$$

which is an equation that yields the externally applied force F (not to be confused with the total force $F_{total} = F + P\pi a^2$) required to achieve a certain stress state. Equations (4.36) and (4.34) form the governing equations of this problem. Filling in (3.11) yields:

$$\begin{cases} P = \int_a^b \left(\lambda_\theta \frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} - \lambda_r \frac{\mathrm{d}W}{\mathrm{d}\lambda_r} \right) \frac{1}{r} \mathrm{d}r \\ F = \int_a^b \left(2\lambda_z \frac{\mathrm{d}W}{\mathrm{d}\lambda_z} - \lambda_r \frac{\mathrm{d}W}{\mathrm{d}\lambda_r} - \lambda_\theta \frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} \right) r \mathrm{d}r \end{cases} . \tag{4.37}$$

To make further analytical progress, we must select a strain energy function.

Neo-Hookian strain energy function The most simple strain energy function, the Neo-Hookian

$$W_{NH} = \frac{C_1}{2} \left(\lambda_\theta^2 + \lambda_z^2 + \lambda_r^2 - 3 \right)$$

can be tried. For this strain energy function we find:

$$\frac{\mathrm{d}W}{\mathrm{d}\lambda_r} = C_1 \lambda_r \qquad \frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} = C_1 \lambda_\theta \qquad \frac{\mathrm{d}W}{\mathrm{d}\lambda_z} = C_1 \lambda_z$$

which we may substitute into the pressure equation (4.37) to find:

$$\begin{aligned} P &= \int_a^b C_1 (\lambda_\theta^2 - \lambda_r^2) \frac{1}{r} \mathrm{d}r \\ &= C_1 \int_a^b \left(\lambda^2 - \frac{1}{\lambda \zeta^2} \right) \frac{1}{r} \mathrm{d}r \\ &= C_1 \int_a^b \left(\frac{r}{R(r)^2} - \frac{1}{\zeta^2} \frac{R(r)^2}{r^3} \right) \mathrm{d}r. \end{aligned}$$

We can then invert (4.30) to find $(R(r))^2 = A^2 + \zeta(r^2 - a^2)$, such that (with help from Mathematica, commands presented in Appendix A) one can find:

$$P = C_1 \left[\frac{A^2 - a^2\zeta}{2r^2\zeta^2} - \frac{\ln(r)}{\zeta} + \frac{\ln(A^2 + \zeta(r^2 - a^2))}{2\zeta} \right]_a^b \tag{4.38}$$

Similarly for F, we may find:

$$\begin{aligned}
F &= \pi \int_a^b C_1 \left(2\lambda_\theta^2 - \lambda_r^2 - \lambda_z^2\right) r \mathrm{d}r \\
&= \pi C_1 \int_a^b 2\left(\zeta^2 - \frac{1}{\lambda^2\zeta^2} - \lambda^2\right) r \mathrm{d}r \\
&= \pi C_1 \left[\left(\zeta^2 - \frac{1}{\zeta^2}\right) r^2 - \frac{A^2 - a^2\zeta}{2\zeta^2}\left(\ln(A^2 + \zeta(r^2 - a^2)) - 2\ln(r)\right)\right]_a^b. \qquad (4.39)
\end{aligned}$$

Mooney Rivlin strain energy function As we have seen the Neo-Hookian strain energy function to be unstable for a cylindrical shell, we may suspect that to be the case for a spherical shell as well. Therefore we try the Mooney-Rivlin strain energy function:

$$W_{MR} = \frac{C_1}{2}(\lambda_r^2 + \lambda_\theta^2 + \lambda_z^2 - 3) + \frac{C_2}{2}(\lambda_r^2\lambda_\theta^2 + \lambda_r^2\lambda_z^2 + \lambda_\theta^2\lambda_z^2 - 3).$$

For which

$$\frac{\mathrm{d}W}{\mathrm{d}\lambda_r} = C_1\lambda_r + C_2\lambda_r(\lambda_\theta^2 + \lambda_z^2) \qquad \frac{\mathrm{d}W}{\mathrm{d}\lambda_\theta} = C_1\lambda_\theta + C_2\lambda_\theta(\lambda_r^2 + \lambda_z^2)$$
$$\frac{\mathrm{d}W}{\mathrm{d}\lambda_z} = C_1\lambda_z + C_2\lambda_z(\lambda_r^2 + \lambda_\theta^2).$$

Substituting these into (4.37) yields:

$$\begin{aligned}
P &= \int_a^b \left(\lambda_\theta^2(C_1 + C_2(\lambda_r^2 + \lambda_z^2)) - \lambda_r^2(C_1 + C_2(\lambda_\theta^2 + \lambda_z^2))\right)\frac{1}{r}\mathrm{d}r \\
&= \int_a^b \left\{\frac{r^3}{R(r)^2}(C_1 + C_2\zeta^2) + \frac{R(r)^2}{r}\left(\frac{-C_1}{\zeta^2} - C_2\right)\right\}\mathrm{d}r \\
&= \frac{a^2\zeta - A^2}{\zeta^2}(C_1 + C_2\zeta^2)\left[\frac{-1}{2r^2} + \frac{\zeta\ln(r)}{A^2 - a^2\zeta} - \frac{\zeta\ln(A^2 + \xi(r^2 - a^2))}{2(A^2 - a^2\zeta)}\right]_a^b \qquad (4.40)
\end{aligned}$$

and

$$\begin{aligned}
F &= \pi\int_a^b (2\lambda_z^2(C_1 + C_2(\lambda_\theta^2 + \lambda_r^2)) - \lambda_\theta^2(C_1 + C_2(\lambda_r^2 + \lambda_z^2)) - \lambda_r^2(C_1 + C_2(\lambda_z^2 + \lambda_\theta^2)))r dr \\
&= \pi\int_a^b \left\{\frac{r^3}{R(r)^2}(C_2\zeta^2 - C_1) + \frac{R(r)^2}{r}\left(C_2 - \frac{C_1}{\zeta^2}\right) + 2r\left(\zeta^2 C_1 - \frac{C_2}{\zeta^2}\right)\right\}\mathrm{d}r \\
&= \frac{\pi}{\zeta^2}\left[r^2(C_2 + C_1\zeta)(\zeta^3 - 1) + (A^2 - a^2\zeta)(C_2\zeta^2 - C_1)\left(\ln(r) - \frac{1}{2}\ln(A^2 + \zeta(r^2 - a^2))\right)\right]_a^b \\
&\qquad (4.41)
\end{aligned}$$

which is not very amenable to further analytical analysis. However, since we have $b = r(B; a)$, we have found explicit expressions for $F(a, \zeta)$ and $P(a, \zeta)$, for both the Neo-Hookian and Mooney-Rivlin strain energy functions.

5 Characteristic pressures of the deformation

Now that we have the explicit functions for P and F, let us start looking for the parameters of Figure 1, namely P_{max} and $P_{propagate}$. Let the force and pressure relationships be written as

$$P = f_1(a, \zeta) \qquad F = f_2(a, \zeta). \tag{5.42}$$

5.1 Finding P_{max}

P_{max} is the maximum pressure at which there is a monotonous increase in strain with pressure, as explained in Section 2. It depends on the material (through C_1 and C_2), the initial shape (A and B), and the applied axial force F.

Analytical solution If we consider F a constant, known input, then we have the following system of three equations for three unknowns:

$$\begin{aligned} 0 &= \frac{\partial f_1(a_{p,max}, \zeta_{p,max})}{\partial a} \\ P_{max} &= f_1(a_{p,max}, \zeta_{p,max}) \\ F &= f_2(a_{p,max}, \zeta_{p,max}) \end{aligned}$$

where we the solution corresponds to lowest value of $a_{p,max}$. Under the assumption that it can be solved explicitly, doing so would provide a solution for P_{max}, parametrised by F.

Numerical solution However, due to the complexity of the equations a numerical technique seems more appropriate, especially if later analysis indicates that the chosen strain energy function is not appropriate for the problem at hand, and a more advanced strain energy function is required to fit the data. To find the numerical solution, we first require the following building block: a 2D root-finding algorithm, that, for a given (F, a), will find (P, ζ) (though we will not be using ζ) for

$$\boldsymbol{g}(F, P, a, \xi) = \begin{bmatrix} F - f_1(a, \xi) \\ P - f_2(a, \xi) \end{bmatrix} = \mathbf{0}. \tag{5.43}$$

Let this algorithm be written as $h_1(F, a) = P$. Then, we apply a maximisation algorithm on this problem,

$$\max_a \; h_1(F, a) = P_{max}, \tag{5.44}$$

yielding the desired result. An initial value of $a = A$ (no deformation), will help ensure the local maximum is found, see Figure 1. As the functions are well behaved, a modified Newton-Raphson (modified to use a finite difference scheme to approximate the derivative(s)) should perform well for a suitable initial guess. As these root-finding algorithms will be run many times for slightly different inputs, it seems sensible to use the solution of the previous step as an initial guess for the current one.

5.2 Finding P_{prop}

To find the propagation pressure, at which the bulge propagates along the axial direction, we need the Maxwell condition, which was modified for the propagation of bulges in cylindrical shells by E. Chater and J.W.H Hutchkinson in [1]. Their derivation is presented here. To find the pressure P_{prop}, we rely on two observations:

1. The change in volume of the balloon, when the transition front moves to engulf a new section that has unit undeformed volume, is $V_D - V_U$ (where V_U is the unit volume of the low strain section and V_D for the high strain section) and the work done hereby is $P_{prop}(V_D - V_U)$. This work done by the pressure is equal to the work performed on a unit of undeformed gas as it passes from state U to D:
$$P_{prop}(V_D - V_U) = \Delta W$$
2. Let $P(V; F)$ be a function that, for a given internal volume V (noting $V = 1 \cdot \pi a^2$) and the total force F, returns the pressure. The isothermal transition of a unit volume of air takes work ΔW, which does not depend on the transition history. In particular, we may calculate ΔW using purely cylindrical deformations to connect states U and D. Hence:
$$\Delta W = \int_{V_U}^{V_D} P(V)dV$$

Combining these two observations, we find the Maxwell condition:

$$P_{prop}(V_D - V_U) = \int_{V_U}^{V_D} P(V)dV \tag{5.45}$$

We can visually interpret the Maxwell condition as specifying the pressure at which the areas I and II, as shown in Figure 7, are equal.

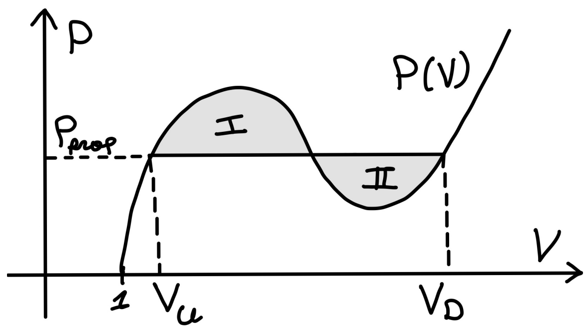

Figure 7: Visualisation of the Maxwell condition

Analytical solution Before we seek a solution for (5.45), we need to find an expression for $P(V;F)$. For this, we may assume unit depth of the section of interest, then $V = 1 \cdot \pi a^2$ such that $a = \sqrt{V/\pi}$. Then, assuming both V and F are known, we have 2 unknowns (P and ζ) and a system of 2 equations:

$$P_{prop} = f_1\left(a = \sqrt{\frac{V}{\pi}}, \zeta\right)$$
$$F = f_2\left(a = \sqrt{\frac{V}{\pi}}, \zeta\right)$$

which could be solved for $P_{prop}(V;F)$. Hereafter we assume F is a known constant such that we have $P_{prop}(V)$. Then, we can find V_D and V_U by solving $P(V) = P_{prop}$, and setting the lowest value equal to V_U, and the highest value to V_D. Then, with explicit expressions for V_U, V_D, and $P(V)$, we should be able to find an explicit expression for P_{prop}.

Numerical solution Just as for P_{max}, a numerical solution might end up being more practical, and hence a solution technique for it is explored here too. We can numerically find $P(V;F)$ by solving (5.43) for (P,ζ), and inputs of $(a = \sqrt{V/\pi}, F)$. Then we can find $V_U(P*)$ and $V_D(P*)$ by finding the roots of $P(V;F) = P*$, and setting the lowest value to V_U and the highest value to V_D. Finally, then (5.45) reduces to a rootfinding problem for $h_2(P*) = P * \cdot (V_D(P*) - V_U(P*)) - \int_{V_U(P*)}^{V_D(P*)} P(\bar{V})\mathrm{d}\bar{V}$, whose solution is then P_{prop}.

6 Conclusion

In this report, we analysed the inflation process of hyperelastic incompressible shells.

We started off by qualitatively explaining the process whereby sections of a hyperelastic shell can be in two different states for the same internal pressure, and how that translates into a complete inflation procedure. Then, we looked at the case of a spherical shell and found an explicit mapping from stretch $\lambda_a = a/A$ to the pressure P for different strain energy functions. This showed how the choice of strain energy function can greatly affect the solution and its validity.

We then proceeded onto the more complicated cylindrical case, which became feasible after some simplifying assumptions. We again found a direct mapping, from the deformation state (ζ, a) to the applied force and pressure (P, F), which is dual due to the extra degree of freedom in the deformation (extension along the radial axis). Finally we outlined how to find P_{max} and P_{prop}, both a theoretical analytical plan and a more feasible numerical technique.

The models and analysis performed in this report provide all the relationships qualitatitively explained in Figure 1 of Section 2: The pressure P vs strain $\lambda_a = a/A$ (and similarly internal volume) relationship, the maximum pressure P_{max} below which there is a monotonous increase in strain for an increase in pressure, and the propagating pressure P_{prop} which is when the bulge propagates towards the low strain section.

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A Mathematica Notebook

Cylindrical Shell - Strain energy function calculations

Setting up the variables

In[2]:=
```
R[r_] := Sqrt[xi * (r ^ 2 - a ^ 2) + A ^ 2]
L[r_] := r / R[r]
lz := xi
lr := 1 / (xi * L[r])
lt := L[r]
b := Sqrt[a ^ 2 + (B ^ 2 - A ^ 2) / xi]

termr := lr ^ 2 (c1 + c2 (lz ^ 2 + lt ^ 2))
termt := lt ^ 2 (c1 + c2 (lz ^ 2 + lr ^ 2))
termz := lz ^ 2 (c1 + c2 (lr ^ 2 + lt ^ 2))
```

Neo-Hookian strain energy function

Pressure relationship

In[25]:=
```
c1 * Integrate[(lt ^ 2 - lr ^ 2) / r, r]
```

Out[25]=

$$c1 \left(-\frac{-A^2 + a^2\, xi}{2\, r^2\, xi^2} - \frac{\text{Log}[r]}{xi} + \frac{\text{Log}[A^2 - a^2\, xi + r^2\, xi]}{2\, xi} \right)$$

Force relationship

In[12]:=
```
ci Pi Integrate[(2 lz ^ 2 - lr ^ 2 - lt ^ 2) r, r]
```

Out[12]=

$$ci\, \pi \left(-\frac{r^2}{xi} + r^2\, xi^2 - \frac{(A^2 - a^2\, xi)\, \text{Log}[r]}{xi^2} - \frac{(-A^2 + a^2\, xi)\, \text{Log}[A^2 - a^2\, xi + r^2\, xi]}{2\, xi^2} \right)$$

Mooney Rivlin strain energy function

Pressure relationship

In[23]:=
```
Integrate[(termt - termr)/r, r]
```

Out[23]=
$$\frac{\left(-A^2+a^2\,xi\right)\left(c1+c2\,xi^2\right)\left(-\frac{1}{2\,r^2}+\frac{xi\,\mathrm{Log}[r]}{A^2-a^2\,xi}-\frac{xi\,\mathrm{Log}[A^2-a^2\,xi+r^2\,xi]}{2\left(A^2-a^2\,xi\right)}\right)}{xi^2}$$

Force relationship

In[24]:=
```
Pi * Integrate[(2 termz - termr - termt) * r, r]
```

Out[24]=
$$\frac{1}{xi^2}\pi\left(r^2\,(c2+c1\,xi)\left(-1+xi^3\right)+\left(A^2-a^2\,xi\right)\left(-c1+c2\,xi^2\right)\mathrm{Log}[r]+\frac{1}{2}\left(A^2-a^2\,xi\right)\left(c1-c2\,xi^2\right)\mathrm{Log}[A^2-a^2\,xi+r^2\,xi]\right)$$

B Code

```
import matplotlib.pyplot as plt
import matplotlib
import numpy as np


font = {'size': 16}
matplotlib.rc('font', **font)
mu = 0.6
A = 1
B = 2
c_1 = 0.621
c_2 = 0.054


def pressure_neo_hookian(lambda_a):
    lambda_b = (1 + A ** 3 * (lambda_a ** 3 - 1) / B ** 3) ** (1 / 3)

    def single(l):
        return -2*mu*(1/l + 1/(4*l**4))

    return single(lambda_a) - single(lambda_b)


def pressure_mooney_rivlin(lambda_a):
    lambda_b = (1 + A ** 3 * (lambda_a ** 3 - 1) / B ** 3) ** (1 / 3)

    def single(l):
        return c_1 * (2*l**(-1) + 0.5*l**(-4)) + c_2 * (l**(-2) - 2*l)

    return single(lambda_b) - single(lambda_a)


lambda_arr = np.arange(1, 10, 0.01)
for func in [pressure_neo_hookian, pressure_mooney_rivlin]:
    plt.plot(lambda_arr, func(lambda_arr), linewidth=2)
    plt.ylabel("Pressure, p [MPa]")
    plt.xlabel(r"Stretch, $\lambda_a$ [-]")
    plt.tight_layout()
    plt.show()
```

plotting_spherical_deformation.py